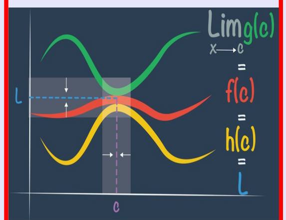
Math 261

Spring 2022

Lecture 12



Class QZ 8

$f(x) = \cos(a^2 + x^3)$

Find $f'(x)$.

No need to Simplify.

$f'(x) = -\sin(a^2 + x^3) \cdot 3x^2$

$= \boxed{-3x^2 \sin(a^2 + x^3)}$

Find $f'(x)$

1) $f(x) = \sqrt{1-2x}$

$f(x) = (1-2x)^{1/2}$

$f'(x) = \frac{1}{2}(1-2x)^{\frac{1}{2}-1} \cdot (-2)$

$= -(1-2x)^{-1/2}$

$$\boxed{f'(x) = \frac{-1}{\sqrt{1-2x}}}$$

2) $f(x) = \sin(x\cos x)$

$f'(x) = \cos(x\cos x) \cdot [1 \cdot \cos x + x \cdot (-\sin x)]$

$$= \boxed{\cos(x\cos x) \cdot [\cos x - x\sin x]}$$

Find $f'(x)$, and simplify

$$f(x) = \frac{(x-1)^4}{(x^2+2x)^5}$$

$$f'(x) = \frac{4(x-1)^3 \cdot 1 \cdot (x^2+2x)^5 - (x-1)^4 \cdot 5(x^2+2x)^4 \cdot \underbrace{(2x+2)}_{2(x+1)}}{[(x^2+2x)^5]^2}$$

$$= \frac{4(x-1)^3(x^2+2x)^5 - 10(x-1)^4(x^2+2x)^4(x+1)}{(x^2+2x)^{10}}$$

$$= \frac{2(x-1)^3\cancel{(x^2+2x)^4}[2(x^2+2x) - 5\overbrace{(x-1)(x+1)}^{x^2-1}]}{\cancel{(x^2+2x)^{10}}}$$

$$= \frac{2(x-1)^3[2x^2+4x-5x^2+5]}{(x^2+2x)^6} = \boxed{\frac{2(x-1)^3(-3x^2+4x+5)}{(x^2+2x)^6}}$$

Find y', do not simplify $y=\left[\cos(\sin x^2)^3\right]^4$

$y=\cos^4(\sin^3 x^2)$

$y' = 4\cos^3(\sin^3 x^2)\cdot -\sin(\sin^3 x^2)\cdot 3\sin^2(x^2)\cdot \cos x^2 \cdot 2x$

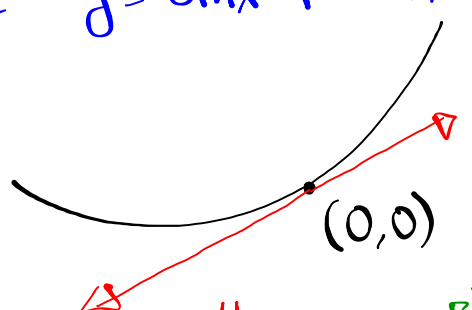
Find y' and y'' of $y=\cos^2 x = [\cos x]^2$

$y' = 2\cos x\cdot -\sin x \Rightarrow y' = -\underbrace{2\sin x\cos x}$

$\boxed{y' = -\sin 2x}$

$y'' = -\cos 2x\cdot 2$ $\boxed{y'' = -2\cos 2x}$

Find eqn of the tangent line to the curve $y=\sin x+\sin^2 x$ at $x=0$.



$y=\sin 0+\sin^2 0$

$=0$

$y'=\cos x+2\sin x\cos x$

$m = y'\big|_{(0,0)} = \overset{1}{\cos 0} + \overset{0}{\cancel{2\cdot\sin 0\cos 0}} = 1$

$y-y_1 = m(x-x_1)$

$y-0 = 1(x-0) \Rightarrow \boxed{y=x}$

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Given $f(1)=2$, $f(2)=3$, $f'(1)=4$, $f'(2)=5$, and $f'(3)=6$. Find $F'(1)$ if $F(x)=f(x\,f(x\,f(x)))$.

$F'(x)=f'(x\,f(x\,f(x)))\cdot\left[1\cdot f(x\,f(x))+x\cdot f'(x\,f(x))\cdot[1\cdot f(x)+x\cdot f'(x)]\right]$

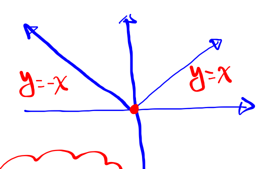
$F'(1)=f'(\underbrace{1\,f(\underbrace{1\,\overbrace{f(1)}^{2}}_{3})}_{3})\cdot\left[\underbrace{f(1\,\overbrace{f(1)}^{2})}_{3}\cdot 1\cdot f'(1\,\overbrace{f(1)}^{2})\cdot[f(1)+1\,f'(1)]\right]$

$=6\cdot[3\cdot5\cdot[2+4]]$

$=6\cdot3\cdot5\cdot6=18\cdot30=\boxed{540}$

Show $\dfrac{d}{dx}[|x|]=\dfrac{x}{|x|}$

$|x|=\begin{cases}-x & \text{if } x<0\\ x & \text{if } x\geq 0\end{cases}$ $\qquad \dfrac{d}{dx}[|x|]=\begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x>0\end{cases}$



$\dfrac{x}{|x|}=\begin{cases}\dfrac{x}{-x}=-1 & \text{if } x<0\\ \dfrac{x}{x}=1 & \text{if } x>0\end{cases}$

$|x|=\sqrt{x^2}$ $\qquad |-5|=\sqrt{(-5)^2}=\sqrt{25}=5$ ✓

$\dfrac{d}{dx}[|x|]=\dfrac{d}{dx}[\sqrt{x^2}]=\dfrac{d}{dx}\left[(x^2)^{\frac{1}{2}}\right]=\dfrac{1}{\cancel{2}}(x^2)^{\frac{1}{2}-1}\cdot\cancel{2}x$

$=(x^2)^{-1/2}\cdot x$

$=\dfrac{x}{(x^2)^{1/2}}$

$=\dfrac{x}{\sqrt{x^2}}=\boxed{\dfrac{x}{|x|}}$

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For n, a positive integer, Prove

$$\frac{d}{dx}\left[\sin^n x \cdot \cos nx\right] = n \boxed{\sin^{n-1} x} \cdot \cos(n+1)x$$

$$\frac{d}{dx}\left[\sin^n x \cos nx\right] = n \sin^{n-1} x \cdot \cos x \cdot \cos nx + \sin^n x \cdot (-\sin nx) \cdot n$$

$$= n \sin^{n-1} x \cdot \cos x \cos nx - n \sin^n x \cdot \sin nx$$

$$= n \left[\boxed{\sin^{n-1} x} \cos x \cos nx - \boxed{\sin^{n-1} x} \sin x \cdot \sin nx\right]$$

$$= n \sin^{n-1} x \left[\cos x \cos nx - \sin x \sin nx\right]$$

$$\underbrace{\cos A \cos B - \sin A \sin B}_{\cos(A+B)}$$

$$= n \sin^{n-1} x \left[\cos(x + nx)\right]$$

$$= \boxed{n \sin^{n-1} x \cos(n+1)x}$$

If θ is measured in degrees, show

$$\frac{d}{d\theta}\left[\sin\theta\right] = \frac{\pi}{180} \cos\theta$$

180 degrees = π radians

1 Degree = $\frac{\pi}{180}$ Radians

$\theta^\circ = \frac{\pi}{180} x$ Radians

$$\frac{d}{dx}\left[\sin \boxed{\frac{\pi}{180} x}\right] = \cos \frac{\pi}{180} x \cdot \frac{\pi}{180} = \frac{\pi}{180} \cos\theta$$

Consider $y = \sqrt{x}$ $\rightarrow$ $y' = \dfrac{1}{2\sqrt{x}}$

Square both sides

$$y^2 = x$$

take derivative of both sides with respect to x

Implicit Differentiation

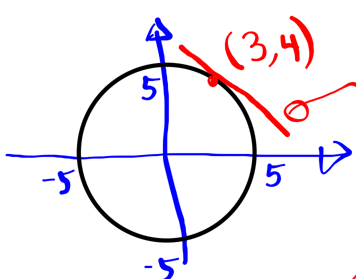
$$\frac{d}{dx}[y^2] = \frac{d}{dx}[x]$$

$$2\cdot y\cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25$

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25]$$



$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 0$$

$$2x + 2y\cdot\frac{dy}{dx} = 0$$

$$m = \frac{dy}{dx}\bigg|_{(3,4)} = \frac{-3}{4}$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \boxed{\frac{-x}{y}}$$

If $x = \sin y$ find $\frac{dy}{dx}$

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

$$1 = \cos y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

Implicit Differentiation

$$\boxed{\frac{dy}{dx} = \sec y}$$

Find $\frac{dy}{dx}$ for $x^2 + xy - y^2 = 4$

$$\frac{d}{dx}\left[x^2 + xy - y^2\right] = \frac{d}{dx}[4]$$

$$2x + 1 \cdot y + \boxed{x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx}} = 0$$

Isolate $\frac{dy}{dx}$

$$\frac{dy}{dx}\left[x - 2y\right] = -2x - y \qquad \boxed{\frac{dy}{dx} = \frac{-2x - y}{x - 2y}}$$

$$\frac{dy}{dx} = \frac{-(2x + y)}{-(2y - x)} \qquad \boxed{\frac{dy}{dx} = \frac{2x + y}{2y - x}}$$

Consider $x^3+y^3=6xy$

1) Show $(3,3)$ is on its graph.

$3^3+3^3=6(3)(3)$ $\quad 27+27=6\cdot 9$

$54=54\checkmark$

2) find $\frac{dy}{dx}$ $\quad \frac{d}{dx}[x^3]+\frac{d}{dx}[y^3]=\frac{d}{dx}[6xy]$

$\cancel{3}x^2+\cancel{3}y^2\cdot\frac{dy}{dx}=\overset{2}{\cancel{6}}\left[1\cdot y+x\cdot\frac{dy}{dx}\right]$

$x^2+y^2\frac{dy}{dx}=2y+\boxed{2x\frac{dy}{dx}}$

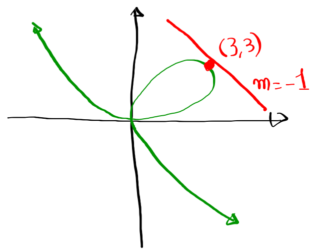
$y^2\frac{dy}{dx}-2x\frac{dy}{dx}=2y-x^2$

3) Evaluate $\left.\frac{dy}{dx}\right|_{(3,3)}$

$(y^2-2x)\frac{dy}{dx}=2y-x^2$

$\left.\frac{dy}{dx}\right|_{(3,3)}=\frac{2\cdot3-3^2}{3^2-2\cdot3}=\boxed{-1}$

$\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$



$y-3=-1(x-3)$

$y-3=-x+3$

$\boxed{y=-x+6}$

find all points where $x^3+y^3=6xy$ has horizontal tan. line. $\rightarrow$ $\frac{dy}{dx}=0$

$m=0$

$f'(x)=0$

$\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$

$y=\frac{x^2}{2}\leftarrow 2y=x^2\leftarrow 2y-x^2=0$ as long as $y^2-2x\neq0$

Plug this into original eqn.

$x^3+y^3=6xy$

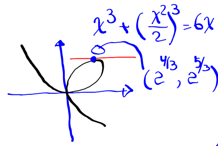
$x^3+\left(\frac{x^2}{2}\right)^3=6x\left(\frac{x^2}{2}\right)$

$x^3+\frac{x^6}{8}=3x^3$

$8x^3+x^6=24x^3$

$x^6-16x^3=0$

$x^3(x^3-16)=0$



$x=0$

$x^3-16=0$

$x=\sqrt[3]{16}$

$=\sqrt[3]{24}$

$\boxed{x=2^{4/3}}$

$x^3+y^3=6xy$

$(\sqrt[3]{16})^3+y^3=6\sqrt[3]{16}\,y$

$y=\frac{x^2}{2}$ $\quad y=\frac{(\sqrt[3]{16})^2}{2}=\frac{(2^{4/3})^2}{2}$

$=\frac{2^{8/3}}{2}=2^{8/3-1}=2^{5/3}$

Point where we have horizontal tan. line is $(2^{4/3},2^{5/3})$

Two curves are called orthogonal if their tangent lines are perpendicular at their intersection points.

$$x^2+y^2=r^2 \Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$

$$ax+by=0 \Rightarrow a+b\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-a}{b}$$

$$\frac{-x}{y}\cdot\frac{-a}{b}=-1 \qquad \frac{ax}{by}=-1$$

$$ax=-by \qquad \frac{-by}{by}=-1 \checkmark$$

Show $y=cx^2$ and $x^2+2y^2=k$ are orthogonal.

$$\frac{dy}{dx}=2cx$$

$$\frac{dy}{dx}\cdot\frac{dy}{dx}=-1$$

$$2x+4y\frac{dy}{dx}=0$$

$$\cancel{2}cx\cdot\frac{-x}{\cancel{2}y}=$$

$$\frac{dy}{dx}=\frac{-x}{2y}$$

$$\frac{-cx^2}{y}=\frac{-y}{y}=-1$$

Find all points on the graph of $x^2y^2 + xy = 2$ where Slope of tan. line is -1.

$$2xy^2 + x^2 \cdot 2y \cdot \frac{dy}{dx} + 1 \cdot y + x \cdot \frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = -1$$

$$\frac{dy}{dx} = \frac{-y - 2xy^2}{2x^2y + x}$$

$$\frac{-y-2xy^2}{2x^2y+x} = -1$$

$$2x^2y + x = y + 2xy^2$$

$$\begin{cases} 2x^2y + x - y - 2xy^2 = 0 \\ x^2y^2 + xy = 2 \end{cases}$$

$$2xy(x-y) + (x-y) = 0$$

$$(x-y)(2xy+1) = 0$$

$x - y = 0$ | $2xy + 1 = 0$

$\boxed{y = x}$ | $y = \frac{-1}{2x}$

$$x^2x^2 + xx = 2$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2-1)(x^2+2) = 0$$

Has no real Solution

$x^2 = 1$

$\boxed{x = \pm 1}$ $\boxed{y = \pm 1}$

$(1,1)$
$(-1,-1)$ Tan. line with slope -1.

$$x^2y^2 + xy = 2 \qquad y = \frac{-1}{2x}$$

$$x^2 \cdot \left(\frac{-1}{2x}\right)^2 + \cancel{x} \cdot \frac{-1}{2\cancel{x}} = 2$$

$$\cancel{x^2} \cdot \frac{1}{4\cancel{x^2}} - \frac{1}{2} = 2$$

$$\frac{1}{4} - \frac{1}{2} = 2 \quad \text{False}$$

No Solution

$x^2y^2 + xy = 2$ has tan. line with Slope -1 at $(1,1)$ & $(-1,-1)$